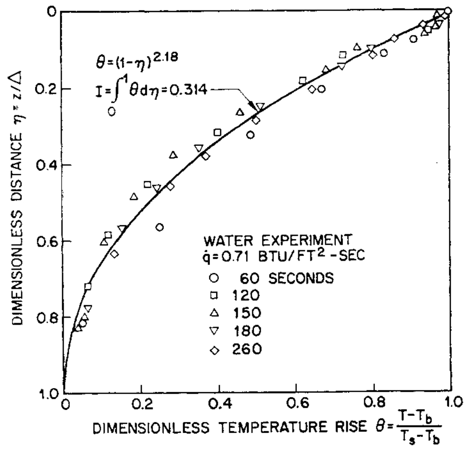

Fig. 3 Typical dimensionless temperature profiles.

values of n ranged from 1.48 to 2.18, resulting in energy integrals I between 0.31 and 0.40. The data shown in Fig. 3 are typical of the nondimensional profiles obtained.

Figure 4 compares the predicted nondimensional temperature rise with the experimental results. The turbulent expression for the stratified layer growth, Eq. (6), was used together with a value of 0.33 for the energy integral. The prediction was made without the evaporation term in the energy equation, since each test was terminated before the liquid surface reached the saturation temperature.

Conclusions

An approximate analysis of the transient stratification of a closed fluid container subjected to constant wall heating has been obtained. The results rely upon the assumption that the dimensionless temperature profiles in the stratified layer do not vary with time. The solution also is restricted to times when the stratified layer thickness is somewhat less than the liquid height. Preliminary experimental data tend to corroborate the analytic results. Further experiments are needed to determine whether or not the dimensionless temperature profiles are always time-invarient, and, if they are, how they are affected by tank size and fluid properties. Until such data are available, the results presented must be considered as tentative; nonetheless, they are useful in pointing out the significant physical and geometric variables governing the phenomenon.

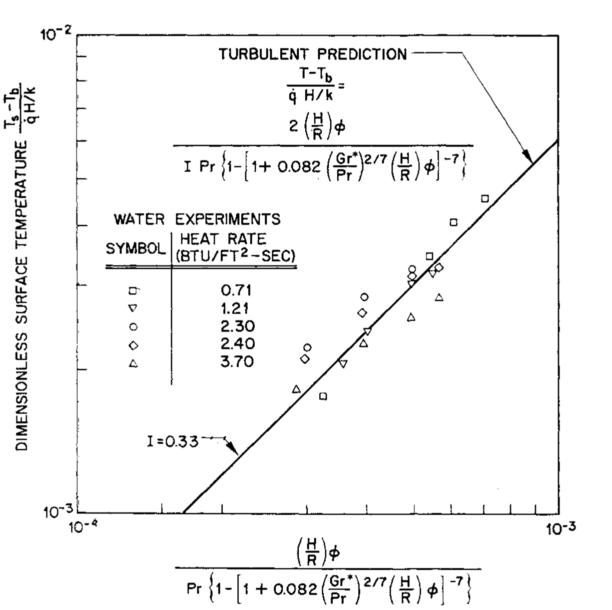

Fig. 4 Comparison of observed and predicted surface temperature rise.

References

[1] Morse, F. H., "Informal notes concerning the natural convection boundary layer in the Rift tank," Lockheed Missiles and Space Co. FM 42 (August 29, 1962).

[2] Eckert, E. R. G. and Jackson, T. W., "Analysis of turbulent free-convection boundary layer on flat plate," NACA Rept. 1015, supersedes NACA TN 2207 (1950).

[3] Sparrow, E. M., "Laminar free convection on a vertical plate with prescribed non-uniform wall heat flux or prescribed non-uniform wall temperature," NACA TN 3508 (July 1955).

Conduction in Thin-Skinned Heat Transfer and Recovery Temperature Models

A. R. George*
Princeton University, Princeton, N. J.

AND

W. G. Reinecke†
Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio

Nomenclature

c = mass specific heat of skin material
E = fractional error due to tangential conduction
h = heat transfer coefficient
k = thermal conductivity of skin material
L = model length
n = harmonic number
q = convective heat transfer rate per unit area
T = skin temperature
T_r = recovery or adiabatic wall temperature
T_i = initial skin temperature
T_{nc} = skin temperature solution neglecting conduction
t = time
t_1, t_2 = restrictions on minimum measuring time as defined in text
α = thermal diffusivity of skin material = $(k/\rho c)$
δ = model skin thickness
δ_0 = scale of the variable skin thickness; (δ/δ_0) is a dimensionless function of the surface coordinates
ρ = density of skin material

THIS note deals with the conduction errors involved in the thin-skinned method of measuring convective heat transfer rates and recovery temperatures. This method uses a wind tunnel model with a thin, usually metal, skin. At the start of an experiment, the model is at a steady, uniform temperature. The gas flow over the model then is established quickly, for example, by injecting the model into the gas stream, and the time variation of the skin temperature is recorded at different positions on the model. The experimenter commonly considers the temperature gradient within the skin normal to the surface to be zero, assumes conduction parallel to the surface and across the inner surface of the skin to be negligible, and thus arrives at the following simple heat balance for an element of skin:

$$h(T_r - T) = q = \rho c \delta (\partial T/\partial t) \tag{1}$$

Then, by measuring $(\partial T/\partial t)$ shortly after the beginning of the heating or by extrapolating backward from later time, he obtains the isothermal heating rate q. Finally, the skin temperature will approach asymptotically its adiabatic or recovery value T_r, if the conduction is again negligible. With q and T_r known, the heat transfer coefficient h may be com-

Received June 17, 1963.

* National Science Foundation Fellow and Graduate Student, Department of Aeronautical Engineering.

† First Lieutenant, U. S. Air Force, Hypersonic Research Laboratory.

puted. Some of the errors introduced into the measured values of q and T_r by the commonly neglected conduction along the model skin now will be analyzed. The analysis will be carried out using a local orthogonal coordinate system on the surface of the model. It is assumed throughout that the inner surface of the skin is perfectly insulated, that the model skin thickness is small with respect to the local principal radii of curvature, and that these radii vary smoothly. However, for some simple geometries the analysis can be made less restrictive, for example, allowing the skin thickness to be a significant fraction of the local radii of curvature.

Consider first the question of temperature uniformity in the direction normal to the skin surface. It is required that $(\alpha t)^{1/2}$, the distance characteristic of the heat diffusion, be much greater than the skin thickness. This criterion is derived in Ref. 1 for a q that is a step function of time and that varies gradually along a uniformly thin skin. The authors of Ref. 1 show that, if $(\pi^2\alpha t/\delta^2) \gg 1$, then the time derivative of skin temperature is constant in the direction normal to the surface. If tangential conduction is negligible, the satisfaction of the foregoing inequality is all that is required to employ Eq. (1). Consideration of the case of a nonsteady q (corresponding to a steady h) shows that it also is necessary that the heat transfer coefficient h be much less than the skin conductance normal to the skin surface (k/δ). If $(h\delta/k)$ is larger than about 0.1, the skin temperature will depart significantly from its initial value before $(\partial T/\partial t)$ becomes constant in the direction normal to the skin surface, and no meaningful value of q can be determined. For the purposes of this note, it will be required further that $(h\delta/k) \ll \frac{1}{10}$ and $(\alpha t/\delta^2) \gg 1$. This will make T itself nearly constant in the direction normal to the skin surface. This may be seen in the tables of Ref. 2.

Now assume that the foregoing two inequalities are satisfied, so that T is nearly uniform normal to the skin surface, and, in addition, assume that $|\nabla\delta| \ll 1$. The heat conduction equation then can be written as

$$\rho c\delta(\partial T/\partial t) = q + \nabla\cdot(\delta k\nabla T) = h(T_r - T) + \delta k\nabla^2 T + k\nabla\delta\cdot\nabla T \quad (2)$$

Now the authors of Ref. 1 obtain the result that for negligible tangential conduction one requires $(\pi^2\alpha t/L^2) \ll 1$, where L is the model length. However, they obtain this result from $(\pi^2 n^2\alpha t/L^2) \ll 1$, where n is the harmonic number in a Fourier expansion of the heat transfer distribution. The assumption is implicit that the higher harmonic coefficients in the expansion can be neglected. This assumption implies that the minimum significant wave length of the expansion of the heat transfer distribution is of the dimension L. This is not the case on bodies that have abrupt changes in heating rates over their surfaces, for example, bodies on which there is boundary-layer separation and reattachment. In these cases it is not clear what characteristic length could be used in the criterion of Ref. 1. In what follows, a more general criterion is derived which allows for an arbitrary distribution of h (and for the resulting nonsteady q) and for a varying δ.

Neglecting conduction normal to the skin resulted in Eq. (2). It now is assumed that the tangential conduction terms $\nabla\cdot(\delta k\nabla T)$ are small compared to the other two terms of the equation. The solution for the temperature distribution with tangential conduction neglected T_{nc} is then

$$T_{nc} = T_r - (T_r - T_i)\exp[-ht/\rho c\delta] \quad (3)$$

E, the fractional error due to tangential conduction, now is defined as the heat gained by an element of skin by conduction divided by the aerodynamic heat input. E is evaluated from T_{nc} and required to be small; that is, it is required that

$$|E| = \left|\frac{\delta k\nabla^2 T_{nc} + k\,\nabla\delta\cdot\nabla T_{nc}}{h(T_r - T_{nc})}\right| \ll 1 \quad (4)$$

This also implies that the method of estimation of E is justified. Then substituting Eq. (3) into the inequality (4) results in

$$|E| = \left|\frac{k}{h(T_r - T_i)}\left[\exp\left(+\frac{t}{\rho c}\frac{h}{\delta}\right) - 1\right] \times (\delta\nabla^2 T_r + \nabla\delta\cdot\nabla T_r) + \frac{2\delta\alpha t}{h(T_r - T_i)}\nabla T_r\cdot\nabla\left(\frac{h}{\delta}\right) + \frac{\alpha t}{h}\nabla\delta\cdot\nabla\left(\frac{h}{\delta}\right) - \frac{\alpha\delta t^2}{\rho c h}\nabla\left(\frac{h}{\delta}\right)\cdot\nabla\left(\frac{h}{\delta}\right) + \frac{\alpha\delta t}{h}\nabla^2\left(\frac{h}{\delta}\right)\right| \ll 1$$

Each of the foregoing terms may be important in some experiments, but if $(ht/\rho c\delta) \ll 1$, then Eq. (3) shows that $|T_{nc} - T_i| \ll |T_r - T_i|$ and thus $h(T_r - T_{nc}) \cong h(T_r - T_i)$. [Note that $(ht/\rho c\delta) \ll 1$, combined with the requirement $(\alpha t/\delta^2) \gg 1$, gives $(h\delta/k) \ll (\delta^2/\alpha t) \ll 1$. This can pose an additional restriction on $(h\delta/k)$, but the restriction often is met in practice.] Under these conditions, T_{nc} may be expanded, and the simplified E will be

$$E \cong \frac{\alpha t}{h(T_r - T_i)}\left\{\delta\nabla^2\left[\frac{h}{\delta}(T_r - T_i)\right] + \nabla\delta\cdot\nabla\left[\frac{h}{\delta}(T_r - T_i)\right]\right\}$$

or, if δ is constant,

$$E \cong \frac{\alpha t}{h(T_r - T_i)}\nabla^2[h(T_r - T_i)]$$

These expressions for E can be used to estimate the conduction error in an experiment, using the empirical values of h and T_r. Moreover, consideration of the variable δ form of E shows that for simple model geometries it may be possible to reduce E by the proper choice of a variable δ. Note that the "limiting solutions" of Ref. 3 are special cases of this simplified E.

Some aspects of the design of experiments now will be considered in light of the foregoing error estimate. The value of E increases with time and is, for small time, proportional to t. Thus, the measurements of $(\partial T/\partial t)$ should be made at the earliest possible time. Now, in general, there are two possible restrictions on the minimum time of measurement. The first is the time required for T to become approximately constant in the direction normal to the skin surface. Call this time $t_1 \cong (\delta^2/\alpha)$. The second restriction on the minimum time is the instrumentation response time or the time required for the establishment of steady flow about the model, whichever is larger. Call this time t_2.

In the case in which t_2 is extremely small, t_1 should be decreased by reducing δ to the minimum possible value consistent with structural design and with negligible heat conduction to the interior of the model or along any attached thermocouple wires (see the analyses in Refs. 4 and 5.) However, in many experiments the earliest possible measurement time is restricted by t_2 rather than by t_1. Under this condition, consider the case of δ constant over the model. (Or in the case of a limited variation of δ, read δ_0 for δ in what follows, where the variable skin thickness δ equals δ_0, a scale thickness, times a dimensionless function of the surface coordinates, this function being everywhere of order one.) If the requirement $(ht/\rho c\delta) \ll 1$ is satisfied, then the simplified E is applicable, and E is completely independent of δ. If the full E is needed, it is seen, after expanding the exponential term of E in an infinite series, that δ appears in E only in some of its terms and then only in their denominators. Thus, in order to reduce these terms, δ should be taken as large as possible without letting t_1 exceed t_2. Moreover, in many practical cases, the full E is only very weakly dependent on δ. The fact that in many cases the error is independent or nearly independent of δ and that in other cases the optimum δ can be quite thick due to a large t_2 generally is not appreciated and is quite important. The use of very thin-skinned models in a vain attempt to reduce conduction errors can lead to structural problems and to errors due to heat conduction to the interior of the model, as noted previously.

The measurement of recovery temperature with thin-skinned models now is considered. The skin temperature at large time approaches a steady state near the recovery temperature. By assuming that this steady T is approximately equal to T_r, one may use T_r to evaluate the conduction term in the steady form of Eq. (2). Thus one obtains

$$T - T_r = (\delta k/h)\nabla^2 T_r + (k/h)\nabla\delta\cdot\nabla T_r$$

This equation can be used to estimate the error in a recovery temperature measurement by using the experimental values of T_r to evaluate the right side. Then the criterion for negligible tangential conduction error in recovery temperature measurements is

$$|(\delta k/h)\nabla^2 T_r + (k/h)\nabla\delta\cdot\nabla T_r| \leq |\text{allowable } T_r \text{ error}|$$

In Ref. 6 a similar result, Eq. (27), is derived more rigorously but is restricted to axisymmetric bodies of uniform skin thickness. Although most investigators use models made of an insulator to measure T_r, if the foregoing criterion is satisfied, then a thin skinned heat transfer model also may be employed to measure T_r.

Each of the errors discussed here can be quite significant in particular experiments. They all should be checked carefully during the design of experiments, particularly when new features such as low or strongly varying heat transfer coefficients are expected.

References

[1] Manos, W. P. and Taylor, D. E., "Analysis of interpretation of data on thin-skinned heat transfer models," J. Heat Transfer **82,** 191–192 (1962).

[2] Smithson, R. E. and Thorne, C. J., "Temperature tables," U. S. Naval Ordnance Test Station NAVORD Rept. 5562 (September 1958).

[3] Conti, R. J., "Approximate temperature distributions and streamwise heat conduction effects in the transient aerodynamic heating of thin-skinned bodies," NASA TN D-895 (September 1961).

[4] Cooper, M. and Mayo, E. E., "Normal conduction effects on heat transfer during transient heating of thin-skin models," J. Aeronaut. Sci. **24,** 461–462 (1957).

[5] McMahon, H. M., "An experimental study of the effect of mass injection at the stagnation point of a blunt body," Graduate Aeronaut. Lab., Calif. Inst. Tech. Hypersonic Research Memo. 42 (May 1958), especially Appendix B.

[6] Sutera, S. P., "Surface temperature distributions on an aerodynamically heated projectile covered with a thin conducting skin," ARS J. **32,** 1421–1424 (1962).

Nonlinear Guidance System for Descent Trajectories

C. N. Shen*

Rensselaer Polytechnic Institute, Troy, N. Y.

A SURVEY paper on space rendezvous, with many references, appeared in *Astronautica Acta.*[1] Terminal guidance for space rendezvous has been divided into two classes: that based on orbital mechanics[2] in the time domain and that based on proportional navigation.[3, 4] This paper deals mainly with problems of descent trajectories including soft-landing.[5–8]

1. General Equations of Motion

A point mass m is attracted to a planet by a central force proportional to the inverse square of the distance. The equations of motion are[9]

$$dk/d\theta = 2a_\theta/u^3 \qquad (1)$$

$$\frac{d^2u}{d\theta^2} + \frac{dk/d\theta}{2k}\frac{du}{d\theta} + u - \frac{g_0}{ku_0^2} = -\frac{a_r}{ku^2} \qquad (2)$$

where

$$u = 1/r = \text{inverse of radius vector} \qquad (3)$$

$$k = h^2 = [r^2\dot{\theta}]^2 = \text{square of specific angular momentum} \qquad (4)$$

and where a_θ, a_r are the transverse and radial specific forces, respectively, and g_0 is the gravitational acceleration at the surface of the planet.

A further transformation is advantageous by virtue of Eq. (1). Let

$$\frac{du}{d\theta} = \frac{dk}{d\theta}\frac{du}{dk} = \left(\frac{2a_\theta}{u^3}\right)\frac{du}{dk} \qquad (5)$$

If Eq. (5) and its derivative with respect to θ are substituted into Eq. (2), one has

$$\frac{d^2u}{dk^2} + \left[\frac{u^3}{2a_\theta}\frac{d}{dk}\left(\frac{2a_\theta}{u^3}\right) + \frac{1}{2k}\right]\frac{du}{dk} + \left(\frac{u^3}{2a_\theta}\right)^2\left[u - \frac{g_0}{ku_0^2} + \frac{a_r}{ku^2}\right] = 0 \qquad (6)$$

and

$$d\theta/dk = u^3/2a_\theta \qquad (7)$$

by inverting Eq. (1).

The dependent variable k in Eq. (1) becomes the independent variable k in Eqs. (6) and (7). Equation (6) is a key equation that does not contain θ explicitly. It is linear in u and can be solved independently of Eq. (7), provided that a_θ/u^3 and a_r/u^2 are expressed in terms of k, such as

$$a_\theta = u^3G_\theta(k) \qquad a_r = u^2G_r(k) \qquad (8)$$

where $G_\theta(k)$ and $G_r(k)$ are functions of k only.

2. Descent Trajectory

A vehicle in orbit is traveling in the direction $b' - b$, as shown on Fig. 1. At point b, the vehicle begins its descent phase by firing a retrorocket having a specific force with a radial component a_r and transverse component a_θ. The descent trajectory is defined by its polar coordinates (r, θ) measured from the point of soft-landing O.

The measurement of θ can be made by inertial means, e.g., an inertial guidance package with both gyros and accelerometers to determine the local vertical with respect to site vertical. The quantity $u = 1/r$ can be determined by measuring the altitude of the vehicle above the planet. The controlled specific forces a_θ and a_r will be in terms of the known quantities θ and u.

For the virgin landing of a vehicle near an unexplored planet, it is desirable to allow the astronaut to hover the vehicle near the surface. The conditions of hovering are zero velocities and zero accelerations in both the transverse and radial directions.

A. Transverse specific force

In order to obtain zero transverse velocity, the specific angular momentum h_0 at point O should be zero; thus, $k_0 = h_0^2 = 0$. The transverse specific force a_θ should be zero so

Received by ARS November 7, 1962; revision received May 27, 1963. The research presented in this paper was supported by NASA under Research Grant No. NsG 14-59.

*Professor of Mechanical Engineering.